In the name of Allah, the Beneficent, the Merciful

ON THE FIELD OF DIFFERENTIAL RATIONAL INVARIANTS OF A SUBGROUP OF AFFINE GROUP(PARTIAL DIFFERENTIAL CASE)

Ural Bekbaev [1]

Department of Mathematics & Institute for Mathematical Research,

FS, UPM, 43400, Serdang, Selangor, Malaysia.

Abstract

An differential field $(F;\partial_1,...,\partial_m)$ of characteristic zero, a subgroup H of affine group $GL(n,C)\propto C^n$ with respect to its identical representation in F^n and the following two fields of differential rational functions in $x=(x_1,x_2,...,x_n)$-column vector,

$$C\langle x,\partial\rangle^H=\{f^{\partial}\langle x\rangle\in C\langle x,\partial\rangle : f^{\partial}\langle hx+h_0\rangle=f^{\partial}\langle x\rangle \text{ whenever } (h,h_0)\in H\},$$

$$C\langle x,\partial\rangle^{(GL^{\partial}(m,F),H)}=\{f^{\partial}\langle x\rangle\in C\langle x,\partial\rangle : f^{g^{-1}\partial}\langle hx+h_0\rangle=f^{\partial}\langle x\rangle \text{ whenever } g\in GL^{\partial}(m,F) \text{ and } (h,h_0)\in H\}$$

are considered, where C is the constant field of (F,∂), $C\langle x,\partial\rangle$ is the field of ∂-differential rational functions in $x_1,x_2,...,x_n$ over C and

$$GL^{\partial}(m,F)=\{g=(g_{jk})_{j,k=\overline{1,m}}\in GL(m,F):\partial_i g_{jk}=\partial_j g_{ik} \text{ for } i,j,k=\overline{1,m}\}$$

, ∂ stands for the column-vector with the "coordinates" $\partial_1,...,\partial_m$. The field $C\langle x,\partial\rangle^H$ ($C\langle x,\partial\rangle^{(GL^{\partial}(m,F),H)}$) is an important tool in the equivalence problem of patches(respect. surfaces) in Differential Geometry with respect to the motion group H. In this paper a pure algebraic approach is offered to describe these fields. The field $C\langle x,\partial\rangle^{(GL^{\partial}(m,F),H)}$ is considered and investigated as a differential field with respect to a commuting system of differential operators $\delta_1,...,\delta_m$. Its relation with differential field $(C\langle x,\partial\rangle^H,\partial)$ is shown. It is shown also that $C\langle x,\partial\rangle^H$ can be derived from some algebraic (without derivatives) invariants of H.

Key words: Differential field, differential rational function, invariant, differential transcendent degree.
2000 Mathematics Subject Classification: 12H05,53A05,53A55

1. Introduction.

Let n, m be natural numbers and H be a subgroup of affine group $GL(n,R)\propto R^n$, $G=Diff(B)$ be the group of diffeomorphisms of the open unit ball $B\subset R^m$, $u:B\to R^n$ be a surface, where u is considered to be infinitely smooth.

A function $f^{\partial}(u(t))$ of $u(t)=(u_1(t),...,u_n(t))$ and its finite number of derivatives relative to $\partial_1=\frac{\partial}{\partial t_1},...,\partial_m=\frac{\partial}{\partial t_m}$ is said to be invariant(more exactly, (G,H)- invariant) if the equality

$$f^{\partial}(u(t))=f^{\delta}(hu(s(t))+h_o)$$

is valid for any $s\in G$, $(h,h_o)\in H$ and $t\in B$, where $u(t)$ stands for the column vector with coordinates $u_1(t),...,u_n(t))$, $s(t)=(s_1(t),...,s_m(t))$, $\delta_i=\frac{\partial}{\partial s_i}$.

Let t run B and $F=C^{\infty}(B,R)$ be the differential ring of infinitely smooth functions relative to differential operators $\partial_1=\frac{\partial}{\partial t_1},...,\partial_m=\frac{\partial}{\partial t_m}$. The constant ring of this differential ring is R i.e.

$$R=\{a\in F:\partial_i a=0 \quad \text{at} \quad i=\overline{1,m}\}$$

[1] e-mail: bekbaev@science.upm.edu.my

Every infinitely smooth surface $u : B \to R^n$ can be considered as an element of differential module $(F^n; \partial_1, \partial_2, ..., \partial_m)$, where $\partial_i = \frac{\partial}{\partial t_i}$ acts on elements of F^n coordinate-wisely. If elements of this module are considered as column vectors the above transformations, involved in definition of invariant function, look like $u = (u_1, ..., u_n) \mapsto hu + h_0, \quad \partial \mapsto g^{-1}\partial$ as far as

$$\frac{\partial}{\partial t_i} = \sum_{j=1}^{m} \frac{\partial s_j(t)}{\partial t_i} \frac{\partial}{\partial s_j(t)}$$

, where g is matrix with the elements $g_{ij} = \frac{\partial s_j(t)}{\partial t_i}$ at $i, j = \overline{1, m}$, ∂ is the column vector with the "coordinates" $\frac{\partial}{\partial t_1}, ..., \frac{\partial}{\partial t_m}$, $(h, h_0) \in H$, $s \in G$. Moreover $\partial_i g_{jk} = \partial_j g_{ik}$ for $i, j, k = \overline{1, m}$.

Therefore for any differential field $(F; \partial_1, \partial_2, ..., \partial_m)$, i.e. F is a field and $\partial_1, \partial_2, ..., \partial_m$ is a given commuting with each other system of differential operators on F, one can consider the transformations

$$u = (u_1, ..., u_n) \mapsto hu + h_0, \qquad \partial \mapsto g^{-1}\partial,$$

, where $u \in F^n$, $(h, h_0) \in H$ is a given subgroup of affine group $GL(n, C) \propto C^n$, $g \in GL^{\partial}(m, F)$,

$$GL^{\partial}(m, F) = \{g \in GL(m, F) : \partial_i g_{jk} = \partial_j g_{ik} \text{ for } i, j, k = \overline{1, m}\}$$

∂ stands for the column-vector with the "coordinates" $\partial_1, ..., \partial_m$, C is the constant field of $(F; \partial)$ i.e.

$$C = \{a \in F : \partial_i a = 0 \quad \text{at} \quad i = \overline{1, m}\}.$$

It should be noted that for any $g \in GL^{\partial}(m, F)$ the differential operators $\delta_1, \delta_2, ..., \delta_m$, where $\delta = g^{-1}\partial$, also commute with each other. So for any $g \in GL^{\partial}(m, F)$ one can consider the differential field (F, δ), where $\delta = g^{-1}\partial$. This transformation is an analogue of gauge transformations for abstract differential field (F, ∂).

Remark 1. In common case the set $GL^{\partial}(m, F)$ is not a group with respect to the ordinary product of matrices as far as it is not closed with respect to that product. But by the use of it a natural groupoid ([1]) can be constructed with the base $\{g^{-1}\partial : g \in GL^{\partial}(m, F)\}$.

Remark 2. Let $g \in GL^{\partial}(m, F)$ and $\delta = g^{-1}\partial$. It is clear that

$$\{a \in F : \partial_1 a = ... = \partial_m a = 0\} = C = \{a \in F : \delta_1 a = ... = \delta_m a = 0\}$$

One interesting question is: When does one have the equality

$$\bigcup_{k \in N} \{a \in F : \partial^{\alpha} a = 0 \text{ whenever } |\alpha| = k\} = \bigcup_{k \in N} \{a \in F : \delta^{\alpha} a = 0 \text{ whenever } |\alpha| = k\} \ ?$$

, where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)$, $|\alpha| = \alpha_1 + \alpha_2 + ...\alpha_m$, α_i are nonnegative integers and $\delta^{\alpha} = \delta_1^{\alpha_1} \delta_2^{\alpha_2} ... \delta_m^{\alpha_m}$.

For example, if $g \in GL^{\partial}(m, F)$, $\det g \in C$ and all entries of g are in

$$\bigcup_{k \in N} \{a \in F : \partial^{\alpha} a = 0 \text{ whenever } |\alpha| = k\}$$

then does it imply the above equality? Of course it is an generalization of the famous Jacobian Conjecture which is the same problem when $F = Q(t_1, ..., t_m)$, $\partial_1 = \frac{\partial}{\partial t_1}$,..., $\partial_m = \frac{\partial}{\partial t_m}$.

Let in future $x_1, ..., x_n$ be differential algebraic independent variables over F and x stand for the column vector with coordinates $x_1, ..., x_n$. We use the following notations : $C[x]$ - the ring of polynomials in $x_1, ..., x_n$ (over C), $C(x)$- the field of rational functions in x, $C\{x, \partial\}$ -the ring of ∂-differential polynomial functions in x and $C\langle x, \partial\rangle$-is the field of ∂-differential rational functions in x over C.

Definition. *An element $f^{\partial}\langle x\rangle \in C\langle x,\partial\rangle$ is said to be $(GL^{\partial}(m,F),H)$- invariant ($GL^{\partial}(m,F)$- invariant; H- invariant) if the equality*

$$f^{g-1\partial}\langle hx+h_0\rangle = f^{\partial}\langle x\rangle$$

(respect. $f^{g-1\partial}\langle x\rangle = f^{\partial}\langle x\rangle; f^{\partial}\langle hx+h_0\rangle = f^{\partial}\langle x\rangle$) is valid for any $g \in GL^{\partial}(m,F)$, $\quad (h,h_0) \in H$.

Let $C\langle x,\partial\rangle^{(GL^{\partial}(m,F),H)}$ ($C\langle x,\partial\rangle^{GL^{\partial}(m,F)}$, $C\langle x,\partial\rangle^{H}$) stand for the set of all such $(GL^{\partial}(m,F),H)$-invariant (respect. $GL^{\partial}(m,F)$- invariant, H- invariant) elements of $C\langle x,\partial\rangle$.

The fields $C\langle x,\partial\rangle^{(GL^{\partial}(m,F),H)}$, $C\langle x,\partial\rangle^{H}$ and their relations are investigated in [2] for the case of $m=1$. Some results on these fields can be found in [3] for the case of $m=n-1$. In the case of finite H more strong results than results of this paper are presented in [4]. The first variant of this paper is published in [5]. The needed notions and results from Differential Algebra can be found in [6].

2. Preliminary

In future let (F,∂) stand for a field F with fixed commuting system of differential operators $\partial_1,...,\partial_m$ and C be its constant field i.e. $C=\{a\in F : \partial_1 a = ... = \partial_m a = 0\}$.

Proposition 1. *If the system of differential operators $\partial_1,...,\partial_m$ is linear independent over F then the differential operators $\delta_1,...,\delta_m$,where $\delta = g^{-1}\partial, g\in GL(m,F)$, commute with each other if and only if $g \in GL^{\partial}(m,F)$.*

Proof. Let $g \in GL(m,F)$, $\delta = g^{-1}\partial$. It is clear that linear independence of $\partial_1,...,\partial_m$ implies linear independence of $\delta_1,...,\delta_m$. For any $i,j=\overline{1,m}$ we have $\partial_j = \sum_{k=1}^{m} g_{jk}\delta_k$, $\partial_i\partial_j = \sum_{k=1}^{m}(\partial_i(g_{jk})\delta_k + g_{jk}\partial_i\delta_k) = \sum_{k=1}^{m}\partial_i(g_{jk})\delta_k + \sum_{k=1}^{m}\sum_{s=1}^{m} g_{jk}g_{is}\delta_s\delta_k$. Therefore due to $\partial_i\partial_j=\partial_j\partial_i$one has

$$\sum_{k=1}^{m}\partial_i(g_{jk})\delta_k + \sum_{k=1}^{m}\sum_{s=1}^{m} g_{jk}g_{is}\delta_s\delta_k = \sum_{k=1}^{m}\partial_j(g_{ik})\delta_k + \sum_{k=1}^{m}\sum_{s=1}^{m} g_{jk}g_{is}\delta_k\delta_s \quad (1)$$

If $\delta_k\delta_s = \delta_s\delta_k$ for any $k,s=\overline{1,m}$ them due (1) one has $\sum_{k=1}^{m}\partial_i(g_{jk})\delta_k = \sum_{k=1}^{m}\partial_j(g_{ik})\delta_k$ i.e. $\partial_i(g_{jk}) = \partial_j(g_{ik})$ for any $i,j,k=\overline{1,m}$ because of linear independence of $\delta_1,...,\delta_m$. Thus in this case $g \in GL^{\partial}(m,F)$.

Vice versa, if $\partial_i(g_{jk}) = \partial_j(g_{ik})$ for any $i,j,k=\overline{1,m}$ then due (1) at any $a\in F$ one has $\sum_{k=1}^{m}\sum_{s=1}^{m} g_{jk}g_{is}\delta_s\delta_k a = \sum_{k=1}^{m}\sum_{s=1}^{m} g_{jk}g_{is}\delta_k\delta_s a$ for any $i,j=\overline{1,m}$. These equalities can be written in the following matrix form $g(\delta_1\delta a,...,\delta_m\delta a)g^t = g(\delta_1\delta a,...,\delta_m\delta a)^t g^t$, where t means transposition. Therefore $(\delta_1\delta a,...,\delta_m\delta a) = (\delta_1\delta a,...,\delta_m\delta a)^t$ i.e. $\delta_k\delta_s a = \delta_s\delta_k a$ for any $k,s=\overline{1,m}$, which completes the proof of Proposition 1.

It should be noted that for $g\in GL^{\partial}(m,F)$ and $\delta = g^{-1}\partial$ the following equality is valid

$$GL^{\delta}(m,F) = g^{-1}GL^{\partial}(m,F). \quad (2)$$

If (K,d) is an ordinary differential field of characteristic zero with a constant field

$$K_0 = \{a\in K : d(a)=0\}$$

then the following criterion is well known: A system of elements $b_1,b_2,...,b_n$ of K is K_0-linear dependent if and only if

$$\det[b, d(b), ..., d^{n-1}(b)] = 0$$

, where b stands for the vector $(b_1,b_2,...,b_n)$. Similar question can be asked in common case: If $b_1,b_2,...,b_n$ is a system of elements of a differential field (F,∂) how one can find out if it is linear dependent over C? The following result deals with this problem in common case.

Consider the differential field (F,∂) of characteristic zero, its constant field C and indeterminates $\{d^i t_j : i\in W, j=\overline{1,m}\}$. Let $F\langle t\rangle$ ($F\{t\}$) stand for the field(respect. ring) of all rational(respect.

polynomial)functions in $\{d^i t_j : i \in W, j = \overline{1,m}\}$ over F. One can make it an ordinary differential field(respect. ring) $F\langle t, d\rangle$ (respect. $F\{t,d\}$) by allowing

1. $d(a) = \sum_{i=1}^{m} \partial_i(a) dt_i$ for any $a \in F$.
2. $d(d^i t_j) = d^{i+1} t_j$ for any $i \in W, j = \overline{1,m}$.

The following result shows that the introduction of the ordinary differential field $F\langle t, d\rangle$ reduces the above stated question once again to the ordinary case.

Proposition 2. *The constant field of the ordinary differential field $F\langle t, d\rangle$ is the came C.*

This result can be deduced easily from the fact that $f^d\{t\}$ divides $df^d\{t\}$ if and only if $f^d\{t\} \in F$, where $f^d\{t\} \in F\{t\}$. As far as $F \subset F\{t\}$ the answer to the above question can be given in the following way: The system $b_1, b_2, ..., b_n$ of elements F is C-linear dependent if and only if $\det[b, d(b), ..., d^{n-1}(b)] = 0$.

Proposition 3. *Let $(F, \partial_1, \partial_2, ..., \partial_m)$- be a differential field of characteristic zero. The following three properties are equivalent.*
a) The system of differential operators $\partial_1, \partial_2, ..., \partial_m$ is linear independent over F.
b) There is no nonzero ∂- differential polynomial over F which vanishes at all values of indeterminates from F.
c) If $p^\partial\{x_{11}, x_{12}, ..., x_{1m}, x_{21}, ..., x_{2m}, ..., x_{m1}, ..., x_{mm}\} = p^\partial\{(x_{ij})_{i,j=\overline{1,m}}\}$ is a differential polynomial over F such that $p^\partial\{g\} = 0$ at any $g = (g_{ij})_{i,j=\overline{1,m}} \in GL^\partial(m, F)$, then $p^\partial\{(t_{ij})_{i,j=\overline{1,m}}\} = 0$ is also valid, for any indeterminates $(t_{ij})_{i,j=\overline{1,m}}$, for which $\partial_k t_{ij} = \partial_i t_{kj}$ at $i, j, k = \overline{1,m}$.

Proof. The equivalence of properties a) and b) is proved in [6, p.139].

It is evident that c) implies b). In fact if there are nonzero ∂- differential polynomials over F which vanish at all values of indeterminates from F we can take one with minimal number of variables. Let $f\{z_1, z_2, ..., z_l\}$ be such a polynomial. If $l > 1$ and $f\{z_1, a_2, ..., a_l\} \neq 0$ for some $a_2, ..., a_l \in F$ then it contradicts minimality of l. Because the nonzero polynomial in one variable $f\{z_1, a_2, ..., a_l\}$ will vanish at all values of z_1 from F. If $l > 1$ and $f\{z_1, a_2, ..., a_l\} = 0$ for all $a_2, ..., a_l \in F$ then considering $f\{z_1, z_2, ..., z_l\}$ as a ∂- differential polynomial in z_1 over $F\{z_2, z_3, ..., z_l\}$ once again we will have a contradiction. Indeed in this case at least one of the coefficients of this polynomial has to be nonzero ∂- differential polynomial in $z_2, z_3, ..., z_l$ (as $f\{z_1, z_2, ..., z_l\}$ is a nonzero polynomial)and vanish at all values of $z_2, z_3, ..., z_l$ from F. It contradicts minimality of l. Thus $l = 1$ and we can consider nonzero polynomial $p^\partial\{(x_{ij})_{i,j=\overline{1,m}}\} = f\{x_{11}\}$ which vanishes at any $g = (g_{ij})_{i,j=\overline{1,m}} \in GL^\partial(m, F)$. This contradicts property c).

Let us prove now that b) implies c). Assume that $p^\partial\{(t_{ij})_{i,j=\overline{1,m}}\} \neq 0$ for some polynomial $p^\partial\{(x_{ij})_{i,j=\overline{1,m}}\}$. Due to the equalities $\partial_k t_{ij} = \partial_i t_{kj},\ i, j, k = \overline{1,m}$ the nonzero $p^\partial\{(t_{ij})_{i,j=\overline{1,m}}\}$ can be represented as a polynomial P of the monomials ${\partial_k}^{n_{k,i}} {\partial_{k+1}}^{n_{k+1,i}} ... {\partial_m}^{n_{m,i}} t_{ki}$, where $n_{j,i}$- are nonnegative integers, $i, k = \overline{1,m}$. Let $t_1, t_2, ..., t_m$ be any differential indeterminates over F. The inequality $0 \neq p^\partial\{(t_{ij})_{i,j=\overline{1,m}}\}$ and substitution $t_{ij} = \partial_i t_j$ give us a nonzero differential polynomial $\det(\partial_i t_j)_{i,j=\overline{1,m}} P$ in $t_1, t_2, ..., t_m$ the value of which at any $(a_1, a_2, ..., a_m)$ from F^m is zero. This contradicts property b).

Proposition 4. *If $a = (a_1, a_2, ..., a_m)$ and $b = (b_1, b_2, ..., b_m)$ are any two nonzero row vectors from F^m then there is such extension (F_1, ∂) of (F, ∂) where the equation $aT = b$ has solution in $GL^\partial(m, F_1)$.*

Proof. Assume , for example, that $a_1 \neq 0$ and $\{t_{ij}\}_{i=\overline{2,m}, j=\overline{1,m}}$ are such differential indeterminates over F that $\partial_k t_{ij} = \partial_i t_{kj}$ for $i, k = \overline{2,m}, j = \overline{1,m}$.

Consider

$$(a_1, a_2, ..., a_m) \begin{pmatrix} y_1 & y_2 & . & . & . & y_m \\ t_{21} & t_{22} & . & . & . & t_{im} \\ . & . & . & . & . & . \\ t_{m1} & t_{m2} & . & . & . & t_{mm} \end{pmatrix} = (b_1, b_2, ..., b_m)$$

as a system of linear equations in $y_1, y_2, ..., y_m$. It has solution

$$(y_1, y_2, ..., y_m) = (t_{11}, t_{12}, ..., t_{1m}) = \frac{1}{a_1}(b - \sum_{i=2}^{m} a_i(t_{i1}, t_{i2}, ..., t_{im}))$$

and the determinant of the corresponding matrix $T = (t_{ij})_{i,j=\overline{1,m}}$ is equal to

$$\frac{1}{a_1} \det \begin{pmatrix} b_1 & b_2 & . & . & . & b_m \\ t_{21} & t_{22} & . & . & . & t_{im} \\ . & . & . & . & . & . \\ t_{m1} & t_{m2} & . & . & . & t_{mm} \end{pmatrix}$$

which is not zero because of $b \neq 0$. Furthermore if one defines $\partial_1(t_{k1}, t_{k2}, ..., t_{km})$ as

$$\partial_1(t_{k1}, t_{k2}, ..., t_{km}) = \partial_k(t_{11}, t_{12}, ..., t_{1m}) = \partial_k(\frac{1}{a_1}(b - \sum_{i=2}^{m} a_i(t_{i1}, t_{i2}, ..., t_{im})))$$

then $T \in GL^{\partial}(m, F_1)$, where $F_1 = F\langle\{t_{i,j}\}_{i=\overline{2,m},j=\overline{1,m}}; \partial\rangle$. This is the proof of Proposition 4.

In future let e stand for the row vector $(1, 0, 0, ..., 0) \in F^m$ and T stand for the matrix $(t_{ij})_{i,j=\overline{1,m}}$, where $\{t_{ij}\}_{i,j=\overline{1,m}}$- are differential indeterminates with the basic relations $\partial_k t_{ij} = \partial_i t_{kj}$ for all $i, j, k = \overline{1,m}$.

Corollary. *If $(F; \partial_1, \partial_2, ..., \partial_m)$- is a differential field of characteristic zero, $\partial_1, \partial_2, ..., \partial_m$ is linear independent over F and $p^{\partial}\{t_1, t_2, ..., t_m, t_{m+1}, ..., t_{m+l}\}$- is an arbitrary nonzero differential polynomial over F then* *a)* $p^{\partial}\{eT, t_{m+1}, ..., t_{m+l}\} \neq 0$,

$$b)\ p^{T^{-1}\partial}\{eT^{-1}, t_{m+1}, ..., t_{m+l}\} \neq 0, \qquad c)\ p^{T^{-1}\partial}\{eT, t_{m+1}, ..., t_{m+l}\} \neq 0.$$

Proof. The proof of inequality a) is evident due to Propositions 3 and 4.

Let us prove b). If one assumes that $p^{T^{-1}\partial}\{eT^{-1}, t_{m+1}, ..., t_{m+l}\} = 0$ then in particular for $T_0 = (\partial_i x_j)_{i,j=\overline{1,m}}$ one has $p^{T_0^{-1}\partial}\{eT_0^{-1}, t_{m+1}, ..., t_{m+l}\} = 0$. Once again it will have to remain be true if one substitutes $T^{-1}\partial$ for ∂ into it. But $\delta_0 = T_0^{-1}\partial$ is invariant with respect to such substitution and T_0 is transformed to $T^{-1}T_0$ so

$$p^{T_0^{-1}\partial}\{eT_0^{-1}T, t_{m+1}, ..., t_{m+l}\} = 0.$$

But for any $S_0 \in GL^{\delta_0}(m, F\langle x_1, ..., x_m; \partial\rangle)$ the equation $T_0^{-1}T = S_0$ has solution in $GL^{\partial}(m, F\langle x_1, ..., x_m; \partial\rangle)$, namely $T = T_0 S_0$. Therefore due to Proposition 3 for the matrix of variables $S = (s_{ij})_{i,j=\overline{1,m}}$ for which $\delta_{0i}s_{jk} = \delta_{0j}s_{ik}$ for all $i, j, k = \overline{1,n}$ one has

$$p^{\delta_0}\{eS, t_{m+1}, ..., t_{m+l}\} = 0.$$

Due to the Corollary, part a), $p^{\delta_0}\{t_1, ..., t_{m+l}\} = 0$ i.e. $p^{\partial}\{t_1, ..., t_{m+l}\} = 0$ which is a contradiction. The proof of c) can be done in a similar way.

3. On $(GL^{\partial}(m, F), H)$- invariants.

In future it is assumed that (F, ∂), where $\partial = (\partial_1, ..., \partial_m)$, is such a differential field that:

1. Characteristic of F is zero.

2. The system $\partial_1, ..., \partial_m$ is linear independent over F.

We use the following obvious fact repeatedly: If $t_1, ..., t_l$ is a ∂-algebraic independent system of variables over F, $g \in GL^{\partial}(m, F)$ and $p^{\partial}\{t_1, ..., t_l\}$ is a ∂-polynomial over F then the following equalities are equivalent.

$$p^{\partial}\{t_1, ..., t_l\} = 0,\ p^{g^{-1}\partial}\{t_1, ..., t_l\} = 0,\ p^{T^{-1}\partial}\{t_1, ..., t_l\} = 0.$$

In future let us assume that for the given subgroup H of $GL(n, C) \propto C^n$ we have such a nonsingular matrix

$$\Phi^{\partial}\langle x\rangle = (\phi^{\partial}_{ij}\langle x\rangle)_{i,j=\overline{1,m}}$$

, where $\phi^{\partial}_{ij}\langle x\rangle \in C\langle x, \partial\rangle$ and $\partial_k \phi^{\partial}_{ij} = \partial_i \phi^{\partial}_{kj}$ for $i, j, k = \overline{1, m}$, that

$$\Phi^{g^{-1}\partial}\langle hx + h_0\rangle = g^{-1}\Phi^{\partial}\langle x\rangle \tag{3}$$

for any $g \in GL^{\partial}(m, F)$ and $(h, h_0) \in H$.

Remark 3. For the given H the existence of the none singular matrix $\Phi^{\partial}\langle x\rangle$ with property (3) is another problem. The paper does not touch this problem. Existence problem of such matrix is considered in [3] in the case of $n = m + 1$.

It is evident that $(C\langle x, \partial\rangle^{H}, \partial)$ is a finitely generated ∂-differential field over C as a subfield of $(C\langle x, \partial\rangle, \partial)$ and $C\langle x, \partial\rangle^{(GL^{\partial}(m,F),H)}$ is a differential field with respect to $\delta = \Phi^{\partial}\langle x\rangle^{-1}\partial$. One of the most important questions is the differential-algebraic transcendence degree of $C\langle x, \partial\rangle^{(GL^{\partial}(m,F),H)}$ as a such δ-field over C.

Theorem 1. *δ-tr.deg.$C\langle x, \partial\rangle^{(GL^{\partial}(m,F),H)}/C = n - m$*

Proof. First of all let us show that the system $\phi^{\partial}_{11}\langle x\rangle, \phi^{\partial}_{12}\langle x\rangle, ..., \phi^{\partial}_{1m}\langle x\rangle$ is δ-algebraic independent over $C\langle x, \partial\rangle^{GL^{\partial}(m,F)}$. Indeed if $p^{\delta}\{t_1, ..., t_m\}$ is such a δ-polynomial over $C\langle x, \partial\rangle^{GL^{\partial}(m,F)}$ for which

$$p^{\delta}\{\phi^{\partial}_{11}\langle x\rangle, \phi^{\partial}_{12}\langle x\rangle, ..., \phi^{\partial}_{1m}\langle x\rangle\} = 0$$

then it will have to remain be true if one substitutes $g^{-1}\partial$ for ∂ into it. Therefore, as far as all coefficients of $p^{\delta}\{t_1, ..., t_m\}$, as well as δ, are invariant with respect to such substitutions and $\Phi^{g^{-1}\partial}\langle x\rangle = g^{-1}\Phi^{\partial}\langle x\rangle$ one has $p^{\delta}\{eg^{-1}\Phi^{\partial}\langle x\rangle\} = 0$ i.e. $p^{\delta}\{eT^{-1}\Phi^{\partial}\langle x\rangle\} = 0$. But for any $S_0 \in GL^{\delta}(m, F\langle x, \partial\rangle)$ the equation $\Phi^{\partial}\langle x\rangle^{-1}T = S_0$ has solution in $GL^{\partial}(m, F\langle x, \partial\rangle)$, namely $T = \Phi^{\partial}\langle x\rangle S_0$. It implies that for the matrix of variables $S = (s_{ij})_{i,j=\overline{1,m}}$, for which $\delta_i s_{jk} = \delta_j s_{ik}$,$i, j, k = \overline{1, m}$, one has $p^{\delta}\{eS^{-1}\} = 0$. Due to Corollary, part a) one has $p^{\delta}\{t_1, ..., t_m\} = 0$.

Now let $f^{\partial}_1\langle x\rangle, ..., f^{\partial}_l\langle x\rangle$ be any system of elements of $C\langle x, \partial\rangle^{GL^{\partial}(m,F)}$. We show that the system

$$\phi^{\partial}_{11}\langle x\rangle, \phi^{\partial}_{12}\langle x\rangle, ..., \phi^{\partial}_{1m}\langle x\rangle, f^{\partial}_1\langle x\rangle, ..., f^{\partial}_l\langle x\rangle$$

is δ-algebraic independent over C if and only if it is ∂-algebraic independent over C.

Indeed if this system is δ-algebraic independent over C and $p^{\partial}\{t_1, ..., t_{m+l}\}$ is any polynomial over C for which

$$p^{\partial}\{e\Phi^{\partial}\langle x\rangle, f^{\partial}_1\langle x\rangle, ..., f^{\partial}_l\langle x\rangle\} = 0$$

then it will have to remain be true if one substitutes $g^{-1}\partial$ for ∂ into it, where $g \in GL^{\partial}(m, F)$. It implies that

$$p^{T^{-1}\partial}\{eT^{-1}\Phi^{\partial}\langle x\rangle, f^{\partial}_1\langle x\rangle, ..., f^{\partial}_l\langle x\rangle\} = 0$$

because $f^{\partial}_1\langle x\rangle, ..., f^{\partial}_l\langle x\rangle$ are invariant with respect to such transformations. But $T^{-1}\partial = (\Phi^{\partial}\langle x\rangle^{-1}T)^{-1}\Phi^{\partial}\langle x\rangle^{-1}\partial = (\Phi^{\partial}\langle x\rangle^{-1}T)^{-1}\delta$ and for any $S_0 \in GL^{\delta}(m, F\langle x; \partial\rangle)$ the equation

$\Phi^\partial\langle x\rangle^{-1}T = S_0$ has solution in $GL^\partial(m, F\langle x,\partial\rangle)$, namely $T = \Phi^\partial\langle x\rangle S_0$. Therefore for the matrix of variables $S = (s_{ij})_{i,j=\overline{1,m}}$, for which $\delta_i s_{jk} = \delta_j s_{ik}$,$i, j, k = \overline{1, m}$, one has

$$p^{S^{-1}\delta}\{eS, f_1^\partial\langle x\rangle, ..., f_l^\partial\langle x\rangle\} = 0.$$

Due to our assumption $f_1^\partial\langle x\rangle, ..., f_l^\partial\langle x\rangle$ is δ-algebraic independent over C and therefore according to Corollary, part b), $p^\delta\{t_1, ..., t_{m+l}\} = 0$ i.e. $p^\partial\{t_1, ..., t_{m+l}\} = 0$. So the system $\phi_{11}^\partial\langle x\rangle, \phi_{12}^\partial\langle x\rangle, ..., \phi_{1m}^\partial\langle x\rangle, f_1^\partial\langle x\rangle, ..., f_l^\partial\langle x\rangle$ has to be ∂-algebraic independent over C.

Vise versa, let $\phi_{11}^\partial\langle x\rangle, \phi_{12}^\partial\langle x\rangle, ..., \phi_{1m}^\partial\langle x\rangle, f_1^\partial\langle x\rangle, ..., f_l^\partial\langle x\rangle$ be ∂-algebraic independent over C. In this case first of all the system $f_1^\partial\langle x\rangle, ..., f_l^\partial\langle x\rangle$ is δ-algebraic independent over C. Indeed $f_1^\partial\langle x\rangle, ..., f_l^\partial\langle x\rangle$ is ∂-algebraic independent over $C\langle\{\phi_{ij}^\partial\langle x\rangle\}_{i,j=\overline{1,m}};\partial\rangle$ as far as $\partial_k\phi_{ij}^\partial\langle x\rangle = \partial_i\phi_{kj}^\partial\langle x\rangle$ for $i, j, k = \overline{1, m}$ and $\phi_{11}^\partial\langle x\rangle, \phi_{12}^\partial\langle x\rangle, ..., \phi_{1m}^\partial\langle x\rangle, f_1^\partial\langle x\rangle, ..., f_l^\partial\langle x\rangle$ is ∂-algebraic independent over C. But every nonzero $p^\delta\{t_1, ..., t_l\}$ over $C\langle\{\phi_{ij}^\partial\langle x\rangle\}_{i,j=\overline{1,m}};\partial\rangle$ can be considered as a nonzero ∂- polynomial over $C\langle\{\phi_{ij}^\partial\langle x\rangle\}_{i,j=\overline{1,m}};\partial\rangle$. Therefore supposition $p^\delta\{f_1^\partial\langle x\rangle, ..., f_l^\partial\langle x\rangle\} = 0$ leads to a contradiction that $f_1^\partial\langle x\rangle, ..., f_l^\partial\langle x\rangle$ is ∂-algebraic independent over $C\langle\{\phi_{ij}^\partial\langle x\rangle\}_{i,j=\overline{1,m}};\partial\rangle$.

Let us assume that for some polynomial $p^\delta\{t_1, ..., t_{m+l}\}$ over C one has $p^\delta\{e\Phi^\partial\langle x\rangle, f_1^\partial\langle x\rangle, ..., f_l^\partial\langle x\rangle\} = 0$. It should remain be true if one substitutes $g^{-1}\partial$ for ∂ into it, where $g \in GL^\partial(m, F)$, which leads to $p^\delta\{eT^{-1}\Phi^\partial\langle x\rangle, f_1^\partial\langle x\rangle, ..., f_l^\partial\langle x\rangle\} = 0$. But the equation $\Phi^\partial\langle x\rangle^{-1}T = S_0$ has solution in $GL^\partial(m, F\langle x,\partial\rangle)$ for any $S_0 \in GL^\delta(m, F\langle x;\partial\rangle)$ and therefore

$$p^\delta\{eS^{-1}, f_1^\partial\langle x\rangle, ..., f_l^\partial\langle x\rangle\} = 0.$$

Now take into consideration that $f_1^\partial\langle x\rangle, ..., f_l^\partial\langle x\rangle$ is δ-algebraic independent over C and Corollary, part a) to see that $p^\delta\{t_1, ..., t_{m+l}\} = 0$. So it is shown that the system

$$\phi_{11}^\partial\langle x\rangle, \phi_{12}^\partial\langle x\rangle, ..., \phi_{1m}^\partial\langle x\rangle, f_1^\partial\langle x\rangle, ..., f_l^\partial\langle x\rangle$$

is δ- algebraic independent over C if and only if it is ∂-algebraic independent over C. In particular it shows that the existence of $\Phi^\partial\langle x\rangle$ with property (3) implies that $m \le n$, because already we have got that the system $\phi_{11}^\partial\langle x\rangle, \phi_{12}^\partial\langle x\rangle, ..., \phi_{1m}^\partial\langle x\rangle$ is δ- algebraic independent over C.

It is evident that the system of components of the matrix $\Phi^\partial\langle x\rangle^{-1} = (\psi_{ij}^\partial\langle x\rangle)_{i,j=\overline{1,m}}$ generates $C\langle x;\partial\rangle$ over $C\langle x,\partial\rangle^{GL^\partial(m,F)}$ as a δ-differential field and $\delta_k\psi_{ij}^\partial\langle x\rangle = \delta_i\psi_{kj}^\partial\langle x\rangle$ for all $i, j, k = \overline{1, m}$, which implies that δ-tr.deg.$C\langle x;\partial\rangle/C\langle x;\partial\rangle^{GL^\partial(m,F)} \le m$. But it already has been established that $\phi_{11}^\partial\langle x\rangle, \phi_{12}^\partial\langle x\rangle, ..., \phi_{1m}^\partial\langle x\rangle$ is δ-algebraic independent over $C\langle x;\partial\rangle^{GL^\partial(m,F)}$ and therefore in reality δ-tr.deg.$C\langle x;\partial\rangle/C\langle x;\partial\rangle^{GL^\partial(m,F)} = m$.

As a ∂-differential field $C\langle x;\partial\rangle$ over C is generated by the elements of $C\langle x;\partial\rangle^{GL^\partial(m,F)}$, as far as $x_1, ..., x_n$ belong to it, and ∂-tr.deg.$C\langle x;\partial\rangle/C = n$. It means that one can find such a system $f_1^\partial\langle x\rangle, ..., f_{n-m}^\partial\langle x\rangle$ elements of $C\langle x;\partial\rangle^{GL^\partial(m,F)}$ for which the system

$$\phi_{11}^\partial\langle x\rangle, \phi_{12}^\partial\langle x\rangle, ..., \phi_{1m}^\partial\langle x\rangle, f_1^\partial\langle x\rangle, ..., f_{n-m}^\partial\langle x\rangle$$

is ∂-algebraic independent over C. As it has been shown that in this case it is δ-algebraic independent over C as well. Therefore δ-tr.deg.$C\langle x;\partial\rangle/C = n$ and δ-tr.deg.$C\langle x;\partial\rangle^{GL^\partial(m,F)}/C = n - m$.

Now to prove Theorem 1 it is enough to show that every $x_1, ..., x_n$ is δ-algebraic over $C\langle x,\partial\rangle^{(GL^\partial(m,F),H)}$. If one assumes that $\det[\delta^{\alpha^1}x, \delta^{\alpha^2}x, ..., \delta^{\alpha^n}x] = 0$ for all nonzero $\alpha^1, \alpha^2, ..., \alpha^n$ from W^n, where W stands for the set of whole numbers, then due to Proposition 2, applied to the differential field $(F\langle x,\partial\rangle, \delta)$, the system $dx_1, dx_2, ..., dx_n$ is linear dependent over C, because of $\det[dx, d^2x, ..., d^nx] = 0$. So there is nontrivial system $c_1, ..., c_n$ elements of C for which $\sum_{i=1}^n c_i dx_i =$

$\sum_{i=1}^{n} c_i dt \cdot \delta x_i = dt \cdot \sum_{i=1}^{n} c_i \delta x_i = dt \cdot \sum_{i=1}^{n} c_i \Phi^{\partial}\langle x\rangle^{-1}\partial x_i = dt \cdot (\Phi^{\partial}\langle x\rangle^{-1}\partial \sum_{i=1}^{n} c_i x_i) = 0$, where dt stands for row vector $(dt_1, dt_2, ..., dt_m)$ and $\cdot$ for the dot product. The last equality implies that $\sum_{i=1}^{n} c_i x_i = 0$, which can occur if and only if $c_1 = c_2 = ... = c_n = 0$. This contradiction shows that one can find nonzero $\alpha^1, \alpha^2, ..., \alpha^n$ from W^n for which $\det[\delta^{\alpha^1}x, \delta^{\alpha^2}x, ..., \delta^{\alpha^n}x] \neq 0$. So now for any nonzero $\alpha \in W^n$ one can consider the following differential equation in y:

$$\det[\delta^{\alpha^1}x, \delta^{\alpha^2}x, ..., \delta^{\alpha^n}x]\det[\delta^{\alpha^1}\overline{x}, \delta^{\alpha^2}\overline{x}, ..., \delta^{\alpha^n}\overline{x}, \delta^{\alpha}\overline{x}] = 0$$

, where $\overline{x} = (x_1, x_2, ..., x_n, y)$, $\delta^{\alpha} = \delta^{(\alpha_1,\alpha_2,...,\alpha_m)}$ stands for $\delta_1^{\alpha_1}\delta_2^{\alpha_2}...\delta_m^{\alpha_m}$. All coefficients of this differential equation belong to $C\langle x, \partial\rangle^{(GL^{\partial}(m,F),H)}$ and $y = x_i$ is a solution for this linear differential equation at any $i = \overline{1,n}$. It implies that indeed δ-tr.deg.$C\langle x, \partial\rangle^{(GL^{\partial}(m,F),H)}/C = n - m$.

The following result says that one can obtain a system of generators of $(C\langle x\rangle^{(GL^{\partial}(m,F),H)}, \delta)$ from the given system of generators of $(C\langle x, \partial\rangle^{H}, \partial)$.

Theorem 2. *If $(C\langle x, \partial\rangle^{H}, \partial)$ as a ∂-differential field over C is generated by a system $(\varphi_i^{\partial}\langle x\rangle)_{i=\overline{1,l}}$ then δ-differential field $(C\langle x, \partial\rangle^{(GL^{\partial}(m,F),H)}, \delta)$ is generated over C by the system $(\varphi_i^{\delta}\langle x\rangle)_{i=\overline{1,l}}$.*

Proof. Let an irreducible $\frac{P^{\partial}\{x\}}{Q^{\partial}\{x\}} \in C\langle x, \partial\rangle^{H}$ be $GL^{\partial}(m, F)$ -invariant. It means that for any $g \in GL^{\partial}(m, F)$ one has the equality

$$P^{g^{-1}\partial}\{x\}Q^{\partial}\{x\} = P^{\partial}\{x\}Q^{g^{-1}\partial}\{x\}$$

Therefore $P^{g^{-1}\partial}\{x\} = P^{\partial}\{x\}\chi^{\partial}\langle g\rangle$. The function $\chi^{\partial}\langle T\rangle$(a "character" of $GL^{\partial}(m, F)$) has the following property

$$\chi^{\partial}\langle g_1 g_2\rangle = \chi^{\partial}\langle g_1\rangle\chi^{g_1^{-1}\partial}\langle g_2\rangle$$

, for any $g_1 \in GL^{\partial}(m, F)$ and $g_2 \in GL^{g_1^{-1}\partial}(m, F)$. But due to (2) one has $g_2 = g_1^{-1}g$ for some $g \in GL^{\partial}(m, F)$ therefore

$$\chi^{\partial}\langle g\rangle = \chi^{\partial}\langle g_1\rangle\chi^{g_1^{-1}\partial}\langle g_1^{-1}g\rangle$$

, for any $g_1, g \in GL^{\partial}(m, F)$. It implies that

$$\chi^{\partial}\langle T\rangle = \chi^{\partial}\langle S\rangle\chi^{S^{-1}\partial}\langle S^{-1}T\rangle$$

, for any $T = (t_{ij})_{i,j=\overline{1,m}}$, $S = (s_{ij})_{i,j=\overline{1,m}}$, for which $\partial_k t_{ij} = \partial_i t_{kj}, \partial_k s_{ij} = \partial_i s_{kj}$ at $i, j, k = \overline{1,m}$. The last equality guarantees that the function $\chi^{\partial}\langle T\rangle$ can not vanish. Therefore $\frac{P^{\partial}\{x\}}{Q^{\partial}\{x\}} = \frac{P^{\delta}\{x\}}{Q^{\delta}\{x\}}$. This is the end of proof Theorem 2.

Let us assume that $(C\langle x, \partial\rangle^{H}, \partial) = C\langle\varphi_1^{\partial}\langle x\rangle, \varphi_2^{\partial}\langle x\rangle, ..., \varphi_l^{\partial}\langle x\rangle, \partial\rangle$. As far as all components of the matrix $\Phi^{\partial}\langle x\rangle$ belong to $C\langle x, \partial\rangle^{H}$ it can be represented in the form

$$\Phi^{\partial}\langle x\rangle = \overline{\Phi}^{\partial}\langle\varphi_1^{\partial}\langle x\rangle, \varphi_2^{\partial}\langle x\rangle, ..., \varphi_l^{\partial}\langle x\rangle\rangle = (\overline{\phi}_{ij}^{\partial}\langle\varphi_1^{\partial}\langle x\rangle, \varphi_2^{\partial}\langle x\rangle, ..., \varphi_l^{\partial}\langle x\rangle\rangle)_{i,j=\overline{1,m}}$$

, where $\overline{\phi}_{ij}^{\partial}\langle t_1, t_2, ..., t_l\rangle \in C\langle t_1, t_2, ..., t_l, \partial\rangle$. Therefore due to $\Phi^{\delta}\langle x\rangle = E_m$ one has

$$\overline{\Phi}^{\delta}\langle\varphi_1^{\delta}\langle x\rangle, \varphi_2^{\delta}\langle x\rangle, ..., \varphi_l^{\delta}\langle x\rangle\rangle = E_m.$$

Remark 4. The equality

$$\chi^{\partial}\langle g\rangle = \chi^{\partial}\langle g_1\rangle\chi^{g_1^{-1}\partial}\langle g_1^{-1}g\rangle$$

, for any $g_1, g \in GL^{\partial}(m, F)$ resembles the property of character of the group $GL(m, F)$. Therefore $\chi^{\partial}\langle T\rangle$ for which the above equality is valid can be considered as a character of the groupoid $GL^{\partial}(m, F)$. Description all such characters is an interesting problem. Of course, $\chi^{\partial}\langle g\rangle = \det(g)^k$,

where k is any integer number, are examples of such characters. Are they all possible differential rational characters of $GL^{\partial}(m,F)$?

Theorem 3. *Any δ-differential polynomial relation over C of the system $\varphi_1^{\delta}\langle x\rangle, \varphi_2^{\delta}\langle x\rangle, ..., \varphi_l^{\delta}\langle x\rangle$ is a consequence of ∂-differential polynomial relations of the system $\varphi_1^{\partial}\langle x\rangle, \varphi_2^{\partial}\langle x\rangle, ..., \varphi_l^{\partial}\langle x\rangle$ over C and the relations $\overline{\Phi}^{\delta}\langle\varphi_1^{\delta}\langle x\rangle, \varphi_2^{\delta}\langle x\rangle, ..., \varphi_l^{\delta}\langle x\rangle\rangle = E_m$.*

Proof. Let $\psi^{\delta}\{\varphi_1^{\delta}\langle x\rangle, \varphi_2^{\delta}\langle x\rangle, ..., \varphi_l^{\delta}\langle x\rangle\} = 0$, where $\psi^{\partial}\{t_1, t_2, ..., t_l\} \in C\{t_1, t_2, ..., t_l, \partial\}$.

If $\psi^{\partial}\{\varphi_1^{\partial}\langle x\rangle, \varphi_2^{\partial}\langle x\rangle, ..., \varphi_l^{\partial}\langle x\rangle\} = 0$ then it means that the above relation ($\psi^{\delta}\{t_1, t_2, ..., t_l\}$) of the system $\varphi_1^{\delta}\langle x\rangle, \varphi_2^{\delta}\langle x\rangle, ..., \varphi_l^{\delta}\langle x\rangle$ is a consequence of the relation ($\psi^{\partial}\{t_1, t_2, ..., t_l\}$) of the system $\varphi_1^{\partial}\langle x\rangle, \varphi_2^{\partial}\langle x\rangle, ..., \varphi_l^{\partial}\langle x\rangle$ i.e. it is obtained by substitution δ for ∂ in $\psi^{\partial}\{t_1, t_2, ..., t_l\}$.

If $\psi^{\partial}\{\varphi_1^{\partial}\langle x\rangle, \varphi_2^{\partial}\langle x\rangle, ..., \varphi_l^{\partial}\langle x\rangle\} \neq 0$ then consider $\psi^{T^{-1}\partial}\{\varphi_1^{T^{-1}\partial}\langle x\rangle, \varphi_2^{T^{-1}\partial}\langle x\rangle, ..., \varphi_l^{T^{-1}\partial}\langle x\rangle\}$ as a ∂-differential rational function in variables $T = (t_{ij})_{i,j=\overline{1,m}}$, where $\partial_k t_{ij} = \partial_i t_{kj}$ for any $i, j, k = \overline{1,m}$ over $C\langle x, \partial\rangle$. Let $\frac{a_x^{\partial}\{T\}}{b_x^{\partial}\{T\}}$ be its irreducible representation and the leading coefficient (with respect to some linear order) of $b_x^{\partial}\{T\}$ be one. We show that in this case all coefficients of $a_x^{\partial}\{T\}, b_x^{\partial}\{T\}$ belong to $C\langle x, \partial\rangle^H$.

Indeed, first of all $\psi^{T^{-1}\partial}\{\varphi_1^{T^{-1}\partial}\langle x\rangle, \varphi_2^{T^{-1}\partial}\langle x\rangle, ..., \varphi_l^{T^{-1}\partial}\langle x\rangle\}$, as a differential rational function in x, is H- invariant function, as much as $\varphi_i^{\partial}\langle x\rangle \in C\langle x, \partial\rangle^H$. This H-invariantness implies that

$$a_x^{\partial}\{T\} b_{hx+h_0}^{\partial}\{T\} = b_x^{\partial}\{T\} a_{hx+h_0}^{\partial}\{T\}$$

for any $(h, h_0) \in H$. Therefore $b_{hx+h_0}^{\partial}\{T\} = \chi^{\partial}\langle x, (h, h_0)\rangle b_x^{\partial}\{T\}$. But comparision of the leading terms of both sides implies that in reality $\chi^{\partial}\langle x, (h, h_0)\rangle = 1$ which in its turn implies that all coefficients of $b_x^{\partial}\{T\}$ (as well as $a_x^{\partial}\{T\}$) are H- invariant.

Therefore all coefficients of $a_x^{\partial}\{T\}$, $b_x^{\partial}\{T\}$ can be considered as ∂-differential rational functions in $\varphi_1^{\partial}\langle x\rangle, \varphi_2^{\partial}\langle x\rangle, ..., \varphi_l^{\partial}\langle x\rangle$, for example, let $b_x^{\partial}\{T\} = \overline{b}_{\varphi_1^{\partial}\langle x\rangle, \varphi_2^{\partial}\langle x\rangle, ..., \varphi_l^{\partial}\langle x\rangle}\{T\}$. Now represent the numerator $a_x^{\partial}\{T\}$ as a ∂-differential polynomial function in $t_{ij} - \overline{\phi}_{ij}^{\partial}\langle\varphi_1^{\partial}\langle x\rangle, \varphi_2^{\partial}\langle x\rangle, ..., \varphi_l^{\partial}\langle x\rangle\rangle$,where $i, j = \overline{1,m}$, for example, let $a_x^{\partial}\{T\} = \overline{a}^{\partial}_{\varphi_1^{\partial}\langle x\rangle, \varphi_2^{\partial}\langle x\rangle, ..., \varphi_l^{\partial}\langle x\rangle}\{T - \overline{\phi}^{\partial}\langle\varphi_1^{\partial}\langle x\rangle, \varphi_2^{\partial}\langle x\rangle, ..., \varphi_l^{\partial}\langle x\rangle\rangle\}$. As such polynomial its constant term is zero because of $\psi^{\delta}\{\varphi_1^{\delta}\langle x\rangle, \varphi_2^{\delta}\langle x\rangle, ..., \varphi_l^{\delta}\langle x\rangle\} = 0$. So

$$\psi^{T^{-1}\partial}\{\varphi_1^{T^{-1}\partial}\langle x\rangle, \varphi_2^{T^{-1}\partial}\langle x\rangle, ..., \varphi_l^{T^{-1}\partial}\langle x\rangle\} = \frac{\overline{a}^{\partial}_{\varphi_1^{\partial}\langle x\rangle, \varphi_2^{\partial}\langle x\rangle, ..., \varphi_l^{\partial}\langle x\rangle}\{T - \overline{\phi}^{\partial}\langle\varphi_1^{\partial}\langle x\rangle, \varphi_2^{\partial}\langle x\rangle, ..., \varphi_l^{\partial}\langle x\rangle\rangle\}}{\overline{b}^{\partial}_{\varphi_1^{\partial}\langle x\rangle, \varphi_2^{\partial}\langle x\rangle, ..., \varphi_l^{\partial}\langle x\rangle}\{T\}}.$$

Substitution $T = E_m$ implies that

$$\psi^{\partial}\{\varphi_1^{\partial}\langle x\rangle, \varphi_2^{\partial}\langle x\rangle, ..., \varphi_l^{\partial}\langle x\rangle\} = \frac{\overline{a}^{\partial}_{\varphi_1^{\partial}\langle x\rangle, \varphi_2^{\partial}\langle x\rangle, ..., \varphi_l^{\partial}\langle x\rangle}\{E_m - \overline{\phi}^{\partial}\langle\varphi_1^{\partial}\langle x\rangle, \varphi_2^{\partial}\langle x\rangle, ..., \varphi_l^{\partial}\langle x\rangle\rangle\}}{\overline{b}^{\partial}_{\varphi_1^{\partial}\langle x\rangle, \varphi_2^{\partial}\langle x\rangle, ..., \varphi_l^{\partial}\langle x\rangle}\{E_m\}}.$$

Now consider the following δ-differential rational function over C:

$$\overline{\psi}^{\delta}\langle t_1, t_2, ..., t_m\rangle = \psi^{\delta}\{t_1, t_2, ..., t_l\} - \frac{\overline{a}^{\delta}_{t_1, t_2, ..., t_l}\{E_m - \overline{\phi}^{\delta}\langle t_1, t_2, ..., t_l\rangle\}}{\overline{b}^{\delta}_{t_1, t_2, ..., t_l}\{E_m\}}.$$

For this function one has

$$\overline{\psi}^{\delta}\langle\varphi_1^{\delta}\langle x\rangle, \varphi_2^{\delta}\langle x\rangle, ..., \varphi_l^{\delta}\langle x\rangle\rangle = 0 \quad \text{as well as} \quad \overline{\psi}^{\partial}\langle\varphi_1^{\partial}\langle x\rangle, \varphi_2^{\partial}\langle x\rangle, ..., \varphi_l^{\partial}\langle x\rangle\rangle = 0.$$

But once again the last equality(relation) means that it is a consequence of relations of the system $\varphi_1^{\partial}\langle x\rangle, \varphi_2^{\partial}\langle x\rangle, ..., \varphi_l^{\partial}\langle x\rangle$. This is the end of proof of Theorem 3.

4. On H- invariants.

The following result provides a method to find generators of the differential field $C\langle x,\partial\rangle^H$ over C. Let $\alpha^1,\alpha^2,...,\alpha^n$ be any different nonzero elements of W^n.

For different classical subgroups H of Affine group the field $C\langle x,\partial\rangle^H$ is investigated in [7] in the case of $m=1$. Our main concern here will be the case of $m\geq 1$ and arbitrary subgroup H of affine group.

Theorem 4. *The equality $C\langle x,\partial\rangle^{GL(n,C)\propto C^n}(x,\partial^{\alpha^1}x,\partial^{\alpha^2}x,...,\partial^{\alpha^n}x)=C\langle x,\partial\rangle$ is valid and moreover the system consisting of components of $x,\partial^{\alpha^1}x,\partial^{\alpha^2}x,...,\partial^{\alpha^n}x$ is algebraic independent over $C\langle x,\partial\rangle^{GL(n,C)\propto C^n}$.*

Proof. For any nonzero $\alpha\in W^n$ consider the differential equation in y:

$$\det[\partial^{\alpha^1}x,\partial^{\alpha^2}x,...,\partial^{\alpha^n}x]\det[\partial^{\alpha^1}\overline{x},\partial^{\alpha^2}\overline{x},...,\partial^{\alpha^n}\overline{x},\partial^{\alpha}\overline{x}]=0$$

, where $\overline{x}$ stands for $(x_1,x_2,...,x_n,y)$. All coefficients of this differential equation are in

$$C\langle x,\partial\rangle^{GL(n,C)\propto C^n}$$

and $y=x_i$ is a solution whenever $i=1,2,...,n$. Therefore

$$C\langle x,\partial\rangle^{GL(n,C)\propto C^n}(x,\partial^{\alpha^1}x,\partial^{\alpha^2}x,...,\partial^{\alpha^n}x)=C\langle x,\partial\rangle$$

To prove algebraic independence of the system $x,\partial^{\alpha^1}x,\partial^{\alpha^2}x,...,\partial^{\alpha^n}x$ over $C\langle x,\partial\rangle^{GL(n,C)\propto C^n}$ it is enough to show algebraic independence of the system $\partial^{\alpha^1}x,\partial^{\alpha^2}x,...,\partial^{\alpha^n}x$ over $C\langle x,\partial\rangle^{GL(n,C)\propto C^n}$.

Let $P[z^1,z^2,...,z^n]$, where $z^i=(z^i_1,z^i_2,...,z^i_n)$, be a nonzero polynomial over $C\langle x,\partial\rangle^{GL(n,C)\propto C^n}$ such that $P[\partial^{\alpha^1}x,\partial^{\alpha^2}x,,...,\partial^{\alpha^n}x]=0$. Assume, for example, at least one of z^i_n, where $i=\overline{1,n}$, occurs in $P[z^1,z^2,...,z^n]$ and

$$P[z^1,z^2,...,z^n]=\sum_\beta(z^1_n)^{\beta_1}(z^2_n)^{\beta_2}...(z^n_n)^{\beta_n}P_\beta[\overline{z^1},\overline{z^2},...,\overline{z^n}]$$

, where $P_\beta[\overline{z^1},\overline{z^2},...,\overline{z^n}]$ are polynomials over $C\langle x,\partial\rangle^{GL(n,C)\propto C^n}$ in $\overline{z^i}=(z^i_1,z^i_2,...,z^i_{n-1})$, $i=\overline{1,n}$.

Consider $h\in GL(n,C)$ which's i-th column is of the form $(0,...,0,1,0,...,0,c_i)$, where $i=\overline{1,n-1}$ and its n-th column is $(0,...,0,c_n)$. For such h one has $\overline{hx}=\overline{x}$. So far as the coefficients of $P[z^1,z^2,...,z^n]$ are $GL(n,C)\propto C^n$- invariant, substitution hx for x into $P[\partial^{\alpha^1}x,\partial^{\alpha^2}x,,...,\partial^{\alpha^n}x]=0$ implies that

$$\sum_\beta(\sum_{i=1}^n c_i\partial^{\alpha^1}x_i)^{\beta_1}(\sum_{i=1}^n c_i\partial^{\alpha^2}x_i)^{\beta_2}...(\sum_{i=1}^n c_i\partial^{\alpha^n}x_i)^{\beta_n}P_\beta[\partial^{\alpha^1}\overline{x},\partial^{\alpha^2}\overline{x},...,\partial^{\alpha^n}\overline{x}]=0.$$

Therefore due to the second assumption on (F,∂) for variables $y_1,y_2,...,y_n$ one has

$$\textstyle\sum_\beta(\sum_{i=1}^n y_i\partial^{\alpha^1}x_i)^{\beta_1}(\sum_{i=1}^n y_i\partial^{\alpha^2}x_i)^{\beta_2}...(\sum_{i=1}^n y_i\partial^{\alpha^n}x_i)^{\beta_n}P_\beta[\partial^{\alpha^1}\overline{x},\partial^{\alpha^2}\overline{x},...,\partial^{\alpha^n}\overline{x}]=0 \qquad (4)$$

Now consider the ring $C\langle x,\partial\rangle[y_1,y_2,...,y_n]$ with respect to the differential operators $\overline{\partial}_1=\frac{\partial}{\partial y_1},\overline{\partial}_2=\frac{\partial}{\partial y_2},...,\overline{\partial}_n=\frac{\partial}{\partial y_n}$. It is clear that its constant ring is $C\langle x,\partial\rangle$ i.e.

$$C\langle x,d\rangle=\{a\in C\langle x,\partial\rangle[y_1,y_2,...,y_n]:\overline{\partial}_1a=\overline{\partial}_2a=...=\overline{\partial}_na=0\}.$$

Introduce new differential operators $\overline{\overline{\partial}}=\sum_{j=1}^n f^\partial_{ij}\langle x\rangle\overline{\partial}_j$, where $i=\overline{1,n}$,

$$(f^\partial_{ij}\langle x\rangle)_{i,j=\overline{1,n}}=[\partial^{\alpha^1}x,\partial^{\alpha^2}x,...,\partial^{\alpha^n}x]^{-1}$$

The following are evident:

a) The constant ring of $(C\langle x,\partial\rangle[y_1,y_2,...,y_n],\overline{\overline{\partial}})$, where $\overline{\overline{\partial}}=(\overline{\overline{\partial}}_1,\overline{\overline{\partial}}_2,...,\overline{\overline{\partial}}_n)$ is the same $C\langle x,\partial\rangle$,

b) $\overline{\overline{\partial}}_j(\sum_{i=1}^n y_i\partial^{\alpha^k}x_i)$ is equal to 0 whenever $j\neq k$ and it is equal to 1 if $j=k$, where $j,k=\overline{1,n}$.

Now if one assumes that $\beta^0=(\beta_1^0,...,\beta_n^0)$ is a such one for which

$$|\beta^0|=\max\{|\beta|:P_\beta[\partial^{\alpha^1}\overline{x},\partial^{\alpha^2}\overline{x},...,\partial^{\alpha^n}\overline{x}]\neq 0\}$$

and applies $\overline{\overline{\partial}}_1^{\beta_1^0}\overline{\overline{\partial}}_2^{\beta_2^0}...\overline{\overline{\partial}}_n^{\beta_n^0}$ to equality (4) he comes to a contradiction $P_{\beta^0}[\partial^{\alpha^1}\overline{x},\partial^{\alpha^2}\overline{x},...,\partial^{\alpha^n}\overline{x}]=0$. This is the end of proof Theorem 4.

So due to Theorem 4

$$C\langle x,\partial\rangle^H=C\langle x,\partial\rangle^{GL(n,C)\propto C^n}(x,\partial^{\alpha^1}x,...,\partial^{\alpha^n}x)^H$$

and the system $x,\partial^{\alpha^1}x,...,\partial^{\alpha^n}x$ is algebraic independent over $C\langle x,\partial\rangle^{GL(n,C)\propto C^n}$. Note that every element of the field $C\langle x,\partial\rangle^{GL(n,C)\propto C^n}$ is a fixed element for the group H. Therefore if one wants to have a system of differential generators of $(C\langle x,\partial\rangle^H,\partial)$ over C he can do the following:

1. Find any system of generators (over C) of the differential field

$$(C\langle x,\partial\rangle^{GL(n,C)\propto C^n},\partial)$$

2. Find any system of ordinary algebraic generators of the field

$$C\langle x,\partial\rangle^{GL(n,C)\propto C^n}(z^1,z^2,...,z^{n+1})^H$$

, where $z^i=(z_1^i,z_2^i,...,z_n^i)$, $i=\overline{1,n+1}$, and the action of H is defined as:

$$((h,h_0),(z^1,z^2,...,z^{n+1}))\to(hz^1+h_0,hz^2,...,hz^{n+1})$$

For example, let it be $\varphi_1(z^1,z^2,...,z^{n+1}),\varphi_2(z^1,z^2,...,z^{n+1}),...,\varphi_k(z^1,z^2,...,z^{n+1})$.

Then the union of the system of generators of $(C\langle x,\partial\rangle^{GL(n,C)\propto C^n},\partial)$ with

$$\{\varphi_1(x,\partial^{\alpha^1}x,...,\partial^{\alpha^n}x),\varphi_2(x,\partial^{\alpha^1}x,...,\partial^{\alpha^n}x),...,\varphi_k(x,\partial^{\alpha^1}x,...,\partial^{\alpha^n}x)\}$$

can be taken as a system of generators of the differential field $(C\langle x,\partial\rangle^H,\partial)$ over C.

In the case of $m=1$ to find a system of generators of the field $C\langle x,\partial\rangle^{GL(n,C)\propto C^n}(z^1,z^2,...,z^{n+1})^H$ it was enough to find generators of $C(z^1,z^2,...,z^{n+1})^H$ due to the fact that the differential field $(C\langle x,\partial\rangle^{GL(n,C)\propto C^n},\partial)$ has a ∂-algebraic independent system of generators over C. It seems that if $m>1$ this fact is not true for $(C\langle x,\partial\rangle^{GL(n,C)\propto C^n},\partial)$ anymore.

Remark 5. At the end I would like to note that a different approach can be done to the equivalence problem of surfaces by the use of rational differential forms. Definition of the ordinary high order differentials of (for example, real) functions of m variables can be changed slightly in such a way that not only the first differential but also all high order differentials will have invariant form with respect to change of variables [8]. One can use it to introduce the field of differential rational forms $R\langle x,d\rangle$, where $x=(x_1,...,x_n)$ is assumed to be variable m-parametric surface. Moreover this field is an ordinary differential field with respect to d. Due to the invariant property of high order differentials with respect to change of variables one have to consider only $R\langle x,d\rangle^H$, where H is a given motion group of R^n. The geometric meaning (or interpretation) of such differential rational forms are not clear but nevertheless one can use results from [2] to find a system of generators of the differential field $(R\langle x,d\rangle^H,d)$.

References

1. A.Weinstein, Groupoids: Unifying Internal and External Symmetry. Notices of the AMS, Volume 43, Number 7, 744-752.

2. Ural Bekbaev, On the field of differential rational invariants of a subgroup of Affine group(Ordinary differential case). arXiv:math: AG/0608479

3. Bekbaev U., Differential rational invariants of Hypersurfaces relative to Affine Group. In Prociding Pengintegrasion Teknologi dalam Sains Matematik, ed. How Guang Aun, Leong Fook, Ong Boon Hua, Quah Soon Hoe, Safian Uda, Zarita Zainiddin, pp. 58-64, Pusat Pengajian Sains Matemarik, USM, 1999.

4. Bekbaev U., On differential rational invariants of finite subgroups of Affine group. Bulletin of Malaysian Mathematical Society, 2005, Volume 28, N1, pp. 55-60.

5. Bekbaev U.D., An algebraic approach to invariants of surfaces. Proceedings of IPTA Research & Development Exposition 2003. 9-12 October 2003. Putra World Trade Center, Kuala Lumpur. Vol. 5: Science and Engineering, ed. Y.A. Khalid et al.,pp. 299-306. University Putra Malaysia Press, Serdang, Selangor, Malaysia.

6. E.R.Kolchin, " Differential Algebra and Algebraic Groups", Academic Press, New York, 1973.

7. Dj.Khadjiev, "Application of Invariant Theory to Differential Geometry of curves", FAN, Tashkent, 1988(Russian).

8. U.D. Bekbaev, High order invariant differentials of functions in local coordinates. Dep.v VINITI, 25.07.90, N4225-B90 (Russian).